MathBootCamps Study Guide

Linear Algebra: Linear Combinations and Span

******Free preview******

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Introduction

This study guide is designed to give you an overview of linear combinations and the span of a set of vectors, along with practice in related computations and in answering theoretical questions. Every question has a full step-by-step solution included at the end of the guide.

Notation and conventions

In linear algebra, notation can sometimes vary. For clarity, here are some quick comments about the notation used in this study guide.

Row equivalence

When two matrices are row equivalent, meaning one is the result of applying a row operation to the other, we use the tilde symbol, ~.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} +R_2 \\ \end{matrix} \sim \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

As you can see above, row operations are shown by placing the operation next to the row.

Vector notation

Vectors are denoted with an arrow over the top. Note that scalars/constants do not include this arrow.

Vector: $\vec{v}$

Scalar: c

Real valued vectors

For all questions and examples, we will only be working with real number constants and vectors from $\mathbb{R}^n$, for some given n.

Here, $\mathbb{R}^n$ represents the set of all vectors with n entries, all of which are real numbers. For example, the following vectors are in $\mathbb{R}^2$:

$$\begin{bmatrix} -10 \\ 5 \end{bmatrix}, \begin{bmatrix} 17 \\ 24 \end{bmatrix}, \begin{bmatrix} 0.1 \\ -6.4 \end{bmatrix}$$

Notice that they each have two entries, both of which are real numbers. Similarly, the following vectors are in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{4} \end{bmatrix}$$

as they each have three real number entries.

Calculators and row reduction

Much of the row reduction in this guide is not shown. It is assumed you will use a calculator or other tool for this, as it is not always necessary to do row operations by hand. You can review the steps for row reduction on the TI83 or TI84 calculator here:

http://www.mathbootcamps.com/row-reduction-ti83-ti84-calculator/

How to use this study guide

If you are working on understanding linear algebra, simply reading through this guide won't be enough. This would be like just reading through a guide about how to play guitar or how to speak a language without trying any of it along the way! To develop real understanding, you should *work* through this guide. That means:

- With a pen and some scratch paper, do the math in each of the examples, trying to understand them as you go.
- Complete each practice problem, and review the solutions before moving on to the next section.
- Look up any unfamiliar terminology.
- Take notes as you read! It may seem strange, but with math, this can be a great way to reinforce ideas.

Understanding linear combinations

Given any number of vectors, a linear combination of the vectors is the result when we multiply each vector by a scalar and add the vectors. For example, if $\vec{u}$ and $\vec{v}$ are vectors, then the following are linear combinations of $\vec{u}$ and $\vec{v}$.

- $3\vec{u}+2\vec{v}$
- $-\vec{u}+0\vec{v}$
- $\dfrac{517}{11}\vec{u}+\dfrac{195}{2}\vec{v}$

Looking at more vectors, say $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ for some $k \geq 1$, a linear combination of the vectors will take the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ for real numbers $c_1, \ldots, c_k$. Again, the idea is to multiply each vector by a real number, and add them up.

Some examples using vectors in $\mathbb{R}^2$:

- $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ since:

  $$\begin{bmatrix} -5 \\ 2 \end{bmatrix} = (1)\begin{bmatrix} -5 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is also a linear combination of $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ since:

  $$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \left(-\frac{1}{5}\right)\begin{bmatrix} -5 \\ 0 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

- $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ since:

  $$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = (5)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{5}{4}\right)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The constants used in a linear combination are often referred to as *weights*. For example, consider the linear combination:

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = (-2)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here, we would say that the vector $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with weights -2 and 3.

The definition of a linear combination also applies to a single vector. If we consider just a single vector $\vec{v}$, then any multiple of $\vec{v}$ is considered a linear combination of $\vec{v}$. For example:

- $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a linear combination of the vector $\begin{bmatrix} -1 \\ \frac{1}{3} \end{bmatrix}$ since:

  $$\begin{bmatrix} -3 \\ 1 \end{bmatrix} = (3)\begin{bmatrix} -1 \\ \frac{1}{3} \end{bmatrix}$$

In the next few practice problems, you will work with the definition of a linear combination. You should especially pay attention to what it may mean to say that a given vector is *not* a linear combination of some set of vectors. This is explored in questions 3 and 4.

Practice 1

1. Find two linear combinations of $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

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